Correspondence Analysis for Visualizing Interplay of Pitch Class, Key, and Composer *

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Abstract

We apply correspondence analysis for visualization of interdependence of pitch class & key and key & composer. A co-occurrence matrix of key & pitch class frequencies is extracted from score (Bach's WTC). Keys are represented as high-dimensional pitch class vectors. Correspondence analysis then projects keys on a planar "keyscape". Vice versa, on "pitchscapes" pitch classes can also be embedded in the key space. In both scenarios a homogenous circle of fifths emerges in the scapes. We employ biplots to embed keys and pitch classes in the keyscape to visualize their interdependence. After a change of co-ordinates the four-dimensional biplots can be interpreted as a configuration on a torus, closely resembling results from music theory and experiments in listener models.

In conjunction with spectral analysis, correspondence analysis constitutes a cognitive auditory model. Correspondence analysis of the co-occurrence table of intensities of keys and pitch classes lets the circle of fifths evolve in the pitchscape. This model works on digitized recorded music, does not require averaging or normalization of the data, and does not implicitly use circularity inherent in the model.

Statistics on key preference in composers yields a composer & key co-occurrence matrix. Then "stylescapes" visualize relations between musical styles of particular composers and schools. The Biplotting technique links stylistic characteristics to favored keys. Interdependence of composers and schools is meaningfully visualized according to their key preferences.

1 Introduction

The correspondence of musical harmony and mathematical beauty has fascinated mankind ever since the Pythagorean idea of the "harmony of the spheres". Of course, there exists a long tradition of analyzing music in mathematical terms. Vice versa, many composers have been inspired by mathematics. In addition, psychophysical experiments have been conducted, e.g., by Krumhansl

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and Kessler (1982) to establish the relation between different major and minor keys in human auditory perception by systematic presentation of Shepard tones. The results of these experiments were visualized by a technique known as *multidimensional scaling* which allows to construct a two-dimensional map of keys with closely related keys close by. As a central result the *circle of fifths* (CoF) as one of the most basic tonal structures could be reproduced. In related work, a self-organizing feature map (Kohonen, 1982) of adaptive artificial neurons was applied to similar data, and showed, how the circle of fifths could be recovered by neural self-organization (Leman, 1995). In Leman and Carreras (1997) cadential chord progressions were embedded in a self-organizing feature map trained on Bach's "Well-Tempered Clavier" (WTC I). Based on that work, a cognitive model consisting of an averaged cq-profile [1] extraction (Purwins et al., 2000a) in combination with a self-organizing feature map revealed the circle of fifths after training on Alfred Cortot's recording of Chopin's Préludes.

In this paper we extend this general idea of embedding musical structure in two-dimensional space by considering the Euclidean embedding of musical entities whose relation is given in terms of a co-occurrence table. This general approach enables us not only to analyze the relation between keys and pitch-classes, but also of other musical entities including aspects of the style of composers. We can, for instance, exploit the fact that composers show strong preferences towards particular keys. This provides the basis for arranging the composers by correspondence analysis reflecting their stylistic relations.

According to Greenacre (1984), the interest in studying co-occurrence tables emerged independently in different fields such as algebra (Hirschfeld, 1935), psychometrics (Horst, 1935; Guttman, 1941), biometrics (Fisher, 1940), and linguistics (Benzécri, 1977). Correspondence analysis was discovered not only in distinct research areas but also in different schools, namely the pragmatic Anglo-American statistical schools as well as the geometric and algebraic French schools. Therefore, various techniques closely related to correspondence analysis have been discussed under various names, e.g., "reciprocal averaging", "optimal (or dual) scaling", "canonical correlation analysis of contingency tables", "simultaneous linear regressions".

We will first introduce the technique of correspondence analysis with a focus on the analysis of co-occurrences of keys and pitch-classes in Section 2. In Section 3 we will present the results of our correspondence analysis of inter-key relations in scores and recorded performances, that leads to the emergence of the circle of fifths and to a toroidal model of inter-key relations. We show how these results relate to a similar model from music theory (Chew, 2000) and to earlier experiments with a different cognitive model (Purwins et al., 2000a). In Section 4 we apply correspondence analysis to the problem of stylistic discrimination of composers based on their key preference. Finally, in Section 5 we point out some relations of our results to previous work and discuss potential application to other analysis tasks arising in music theory. Please note that we provide a more technical perspective on correspondence analysis in the Appendix, Section 6.

1 The abbreviation *cq* refers to *constant Q*, denoting a transformation with uniform resolution in the logarithmic frequency domain with a resulting constant ratio between frequency and band-width.

2 Analysis of Co-occurrence

Co-occurrence data frequently arise in various fields ranging from the co-occurrences of words in documents (information retrieval) to the co-occurrence of goods in shopping baskets (data mining). In the more general case, we consider the co-occurrence of two different features. One feature $\mathcal{K}$ is described by a vector that contains the frequencies how often it co-occurs with each specification of the other feature $\mathcal{P}$ and vice versa. Correspondence analysis aims at embedding the features $\mathcal{K}$ in a lower-dimensional space such that the spatial relations in that space display the similarity of the features $\mathcal{K}$ as reflected by their co-occurrences together with feature $\mathcal{P}$.

Co-occurrence Table. Consider, as our running example, the co-occurrence table

$$\mathbf{H}^{\mathcal{K},\mathcal{P}} = (h_{ij}^{\mathcal{K},\mathcal{P}})_{\substack{1\leq i\leq 24\\ 1\leq j\leq 12}} \tag{1}$$

for keys ($\mathcal{K}$) and pitch classes ($\mathcal{P}$).

	c	...	...	b	$\mathbf{h}^{\mathcal{K}}$
C	$h_{\mathrm{C,c}}^{\mathcal{K},\mathcal{P}}$	...		$h_{\mathrm{C,b}}^{\mathcal{K},\mathcal{P}}$	$h_{\mathrm{C}}^{\mathcal{K}}$
...	...				...
B	...				$h_{\mathrm{B}}^{\mathcal{K}}$
Cm	...				$h_{\mathrm{Cm}}^{\mathcal{K}}$
...	...				...
Bm	$h_{\mathrm{Bm,c}}^{\mathcal{K},\mathcal{P}}$	...		$h_{\mathrm{Bm,b}}^{\mathcal{K},\mathcal{P}}$	$h_{\mathrm{Bm}}^{\mathcal{K}}$
$\mathbf{h}^{\mathcal{P}}$	$h_{\mathrm{c}}^{\mathcal{P}}$	...		$h_{\mathrm{b}}^{\mathcal{P}}$	n

Table $\mathbf{H}^{\mathcal{K},\mathcal{P}}$ reflects the relation between two sets $\mathcal{K}$ and $\mathcal{P}$ of features or events (cf. Greenacre (1984)), in our case $\mathcal{K} = \{\mathrm{C}, \ldots, \mathrm{B}, \mathrm{Cm}, \ldots, \mathrm{Bm}\}$ being the set of different keys, and $\mathcal{P} = \{\mathrm{c}, \ldots, \mathrm{b}\}$ being the set of different pitch classes. Then an entry $h_{ij}^{\mathcal{K},\mathcal{P}}$ in the co-occurrence table would just be the number of occurrences of a particular pitch class $j \in \mathcal{P}$ in musical pieces of key $i \in \mathcal{K}$. The frequency $h_i^{\mathcal{K}}$ is the summation of occurrences of key i across all pitch classes. The frequency of pitch class j accumulated across all keys is denoted by $h_j^{\mathcal{P}}$. The sum of the occurrences of all pitch classes in all keys is denoted by n. From a co-occurrence table one can expect to gain information about both sets of features, $\mathcal{K}$ and $\mathcal{P}$, and about the relation between features in $\mathcal{K}$ and $\mathcal{P}$, i.e., between keys and pitch classes in the example above. The relative frequency of the entries is denoted by

$$f_{ij}^{\mathcal{K},\mathcal{P}} = \frac{1}{n} h_{ij}^{\mathcal{K},\mathcal{P}} . \tag{2}$$

It is the joint distribution of $\mathcal{K} \times \mathcal{P}$. The relative frequency of column j is $f_j^{\mathcal{P}} = \frac{1}{n} h_j^{\mathcal{P}}$. It is the marginal distribution of $f_{ij}^{\mathcal{K},\mathcal{P}}$. The diagonal matrix with

$\mathbf{f}^{\mathcal{P}}$ on the diagonal is denoted $\mathbf{F}^{\mathcal{P},\mathcal{P}}$. The conditional relative frequency is denoted by

$$f_j^{\mathcal{P}|\mathcal{K}=i} = \frac{h_{ij}^{\mathcal{K},\mathcal{P}}}{h_i^{\mathcal{K}}}\,, \tag{3}$$

in matrix notation: $\mathbf{F}^{\mathcal{P}|\mathcal{K}} = (f_j^{\mathcal{P}|\mathcal{K}=\mathrm{i}})_{ji}$.

Instead of co-occurrence tables $\mathbf{H}^{\mathcal{K},\mathcal{P}}$ of frequencies of occurrences, in the sequel, we will also consider co-occurrence tables of overall symbolic durations (cf. Section 3.1) as well as co-occurrence tables of accumulated intensities (cf. Section 3.2).

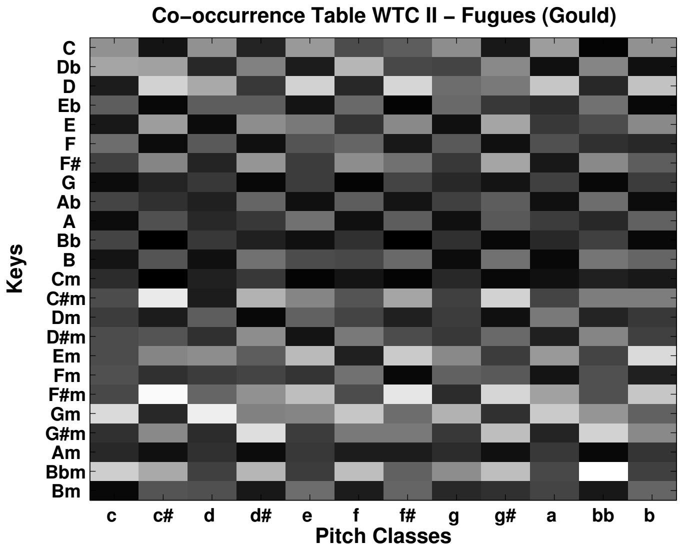

Figure 1: Co-occurrence table (cf. Table) of Bach's "Well-Tempered Clavier", Fugues of Book II recorded by Glenn Gould. The keys of the 24 Fugues are labeled on the vertical axis. For each fugue the intensities are accumulated for each pitch class, calculating cq-profiles (Purwins et al., 2000b). Light color indicates high intensity. Dark color indicates low intensity. This table is analyzed in Section 3.2.

2.1 Correspondence Analysis

Given a co-occurrence table $\mathbf{H}^{\mathcal{K},\mathcal{P}}$, for visualization purposes we aim at displaying the features $\mathcal{K}$ and $\mathcal{P}$ in a two-dimensional space, such that aspects of their tonal relation are reflected by their spatial configuration. In particular, correspondence analysis can be thought of as a method that aims at finding a new co-ordinate system that optimally preserves the χ^2-distance between the

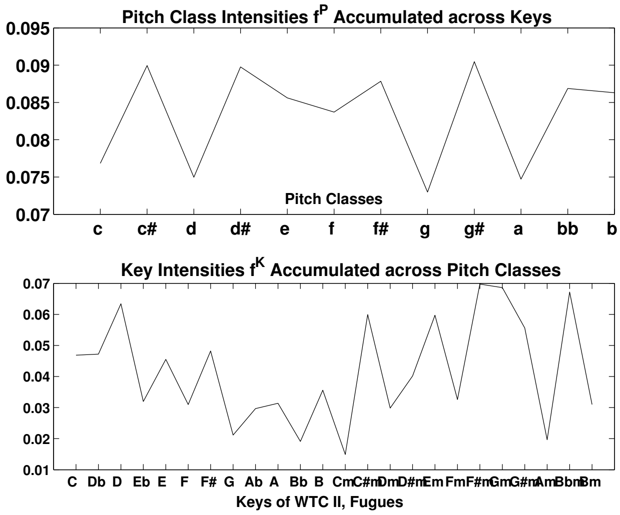

Figure 2: Relative frequency of pitch classes $\mathbf{f}^{\mathcal{P}}$ and keys $\mathbf{f}^{\mathcal{K}}$ of performed WTC II, Fugues, accumulated from the co-occurrence table (Figure 1). It is remarkable that the non-diatonic notes in C–Major are the most prominent notes, as if Bach would have wanted to oppose to the emphasis of C–Major in mean tone tuning. *Upper:* $\mathbf{f}^{\mathcal{P}}$ is the normalized vector of pitch class intensities accumulated across all fugues in WTC II. *Lower:* $\mathbf{f}^{\mathcal{K}}$ is the normalized vector of accumulated intensity of each fugue.

frequency $\mathcal{K}$-, and $\mathcal{P}$-profiles, i.e., of columns and rows. For 12-dimensional pitch class frequency vectors $\mathbf{a}$ and $\mathbf{b}$ the χ^2-distance is defined by

$$\|\mathbf{a}-\mathbf{b}\|^2_{\mathcal{P}} := \langle \mathbf{a},\mathbf{a}\rangle_{\mathcal{P}} - 2\langle \mathbf{a},\mathbf{b}\rangle_{\mathcal{P}} + \langle \mathbf{b},\mathbf{b}\rangle_{\mathcal{P}} \tag{4}$$

with a generalized inner product defined by

$$\langle \mathbf{a},\mathbf{b}\rangle_{\mathcal{P}} := \mathbf{a}'(\mathbf{F}^{\mathcal{P},\mathcal{P}})^{-1}\mathbf{b}\,, \tag{5}$$

where $\mathbf{a}'$ denotes the transpose of vector $\mathbf{a}$. The χ^2-distance is equal to the Euclidean distance in this example if all pitch classes appear equally often. The χ^2-distance weights the components by the overall frequency of occurrence of pitch classes, i.e., rare pitch classes have a lower weight than more frequent pitch classes. The χ^2-distance satisfies the natural requirement that pooling subsets of columns into a single column, respectively, does not distort the overall embedding because the new column carries the combined weights of its constituents. The same holds for rows.

We can explain correspondence analysis by a comparison to principal component analysis. In principal component analysis eigenvalue decomposition is used to rotate the co-ordinate system to a new one with the axes given by the eigenvectors. The eigenvalue associated with each eigenvector quantifies the prominence of the contribution of this particular co-ordinate for explaining the variance of the data. The eigenvector with highest eigenvalue indicates the most important axis in the data space: the axis with highest projected variance. Visualization in this framework amounts to projecting the high-dimensional data (the 12-dimensional pitch class frequency space or, respectively, the 24-dimensional key frequency space) onto a small number (typically 2 or 3) of eigenvectors with high eigenvalues. Hereby only insignificant dimensions of the data space are discarded, leading, effectively, to a plot of high-dimensional data in 2d or 3d space.

In principal component analysis by rotating the co-ordinate system, the Euclidean distances between data points are preserved. Correspondence analysis is a generalization of principal component analysis: The χ^2 distance (a generalization of the Euclidean distance) between data points is preserved.

If the data matrix is not singular and not even symmetric, generalized singular value decomposition instead of eigenvalue decomposition yields two sets of *factors* $\mathbf{u}_1, \ldots, \mathbf{u}_d$ and $\mathbf{v}_1, \ldots, \mathbf{v}_d$ instead of one set of eigenvectors. So either for the m-dimensional column vectors of the data matrix the co-ordinate system can be rotated yielding a new co-ordinate system given by the column factors $\mathbf{u}_1, \ldots, \mathbf{u}_d$, or the n-dimensional row vectors of the data matrix are expressed in terms of co-ordinates in the new co-ordinate system of row factors $\mathbf{v}_1, \ldots, \mathbf{v}_d$. In principal component analysis each eigenvector is associated with an eigenvalue. In the same sense for each pair of column and row vectors $\mathbf{u}_k$ and $\mathbf{v}_k$, an associated singular value δ_{kk} quantifies the amount of variance explained by these factors (see the appendix, Section 6 for technical details). Consider the conditional relative frequency of pitch classes $\mathbf{F}^{\mathcal{P}|\mathcal{K}}$ being the data matrix. If we project the 12-dimensional pitch class profile $\mathbf{f}^{\,\mathcal{P}|\mathcal{K}=i}$ into the space spanned by all d vectors $\mathbf{v}_1, \ldots, \mathbf{v}_d$ and represent each profile $\mathbf{f}^{\,\mathcal{P}|\mathcal{K}=i}$ by its d-dimensional co-ordinate vector $\mathbf{s}_i$, then the χ^2-distance between $\mathbf{f}^{\,\mathcal{P}|\mathcal{K}=i}$ and $\mathbf{f}^{\,\mathcal{P}|\mathcal{K}=l}$ equals the Euclidian distance between the co-ordinate vectors $\mathbf{s}_i$ and $\mathbf{s}_l$ of their projections. But if we only use the two co-ordinates with highest singular value, instead of all d co-ordinates, then all distances are contracted and more or less distorted, depending on the singular values.

A *biplot* provides a simultaneous projection of features $\mathcal{K}$ and $\mathcal{P}$ into the same space. Both the co-ordinates of a $\mathcal{K}$-profile in the co-ordinate system of the $\mathbf{u}_k$'s and the co-ordinates of a $\mathcal{P}$-profile in the co-ordinate system of the $\mathbf{v}_k$'s are displayed in the same co-ordinate system. Such a biplot may reveal the inter-set relationships.

3 Circle of Fifths in the Keyscape

We will now investigate the set of Preludes & Fugues in Bach's WTC. For each part of WTC there is a one-to-one mapping between all 24 pairs of Preludes & Fugues and all 24 Major and minor keys. The Table above shows how each key - that implies each Prelude & Fugue pair also - can be represented by a frequency profile of pitch classes. The pitch class frequency profiles can either contain the overall symbolic durations from the score or the accumulated cq-profiles from a performance of that piece. Correspondence analysis visualizes inter-key relations on keyscapes based on pitch class profiles. The projection of pitch classes homogeneously displays the circle of fifths for both score and performance (Sections 3.1 and 3.2). The display of phases between projections of the pitch class profiles from score can be interpreted as a regular Toroidal

Model of Inter-Key Relations (Section 3.1), consistent with other models (Purwins et al., 2000a; Chew, 2000).

3.1 Circle of Fifths from Score

Humdrum (CCARH, 2003) is a digital format that is aimed at fully and precisely representing the essential content of a composition as it is obvious from notation in a score. All fugues of Bach's WTC are encoded in Humdrum **kern format. Instead of analyzing the co-occurrence table of frequencies of keys and pitch classes we look at the *overall symbolic duration* of pieces in a particular key and the overall symbolic duration of pitch classes across all 24 Fugues for WTC I and for WTC II. Symbolic duration means that it is measured in multiples and fractions of quarter notes, rather than in seconds. Measuring duration in seconds would imply that the results would vary a lot depending on the choice of tempo. But the issue of tempo is highly discussed in Bach.

The correspondence analysis of WTC (Figure 3) reveals a two–dimensional structure which allows for an adequate representation in a plot based on the first two factors corresponding to the two largest singular values (Figure 5). The 24 pitch class frequency profiles are optimally projected onto a 2-dimensional plane, such that the χ^2-distance between profiles is minimally distorted and the χ^2-distance of the 2-dimensional projections matches the original profile distance as well as possible. In the Fugues of WTC, the circle of fifths emerges clearly and homogeneously (upper Figure 3). Even though in the upper Figure 3 some inter-key distances are smaller (D♭–A♭) than others (D–A), due to different χ^2-distances between pitch class prominence profiles in these pieces. In addition the minor keys form a circle of fifths inside the circle of fifths of the major keys. This shows that the pitch prominence patterns for major keys are more distinct than for minor keys according to the metric employed.

In the co-occurrence table above pitch classes are represented as columns of accumulated symbolic durations in the different keys, that means accumulated symbolic durations in each fugue of WTC I, since in WTC I a one-to-one correspondence of Fugues and the 24 keys is given. The same factors with maximal singular values as in the upper Figure 3 are used to optimally project the 24-dimensional pitch class vectors upon a plane. We observe the pitch classes forming a circle of fifths as well (middle Figure 3). We can now consider the biplot (lower Figure 3) by putting the transparent plot of pitch classes (middle Figure 3) on top of the plot of keys (upper Figure 3). We have three circles of fifths, one each for the Major and minor keys and one for the pitch classes. We change the co-ordinates of the factor plane to polar co-ordinates in terms of a polar angle (on the circle of fifths) and the distance to the origin. Consider the angles of both the Major and minor keys relative to the angles of the fundamental pitch class (Figure 4). The plot shows two almost straight parallel lines, reflecting that pitch classes and keys proceed through the circle of fifths with almost constant offset. The graph for the minor keys is almost the identity, indicating that pitch classes and keys are "in phase": The pitch classes can be identified with the fundamentals of the minor keys. The explanation lies in the relatively high overall symbolic duration of the fundamental pitch class in the symbolic duration profile in minor compared to Major. Also the overall symbolic duration of the Fugues in minor is longer than the one in

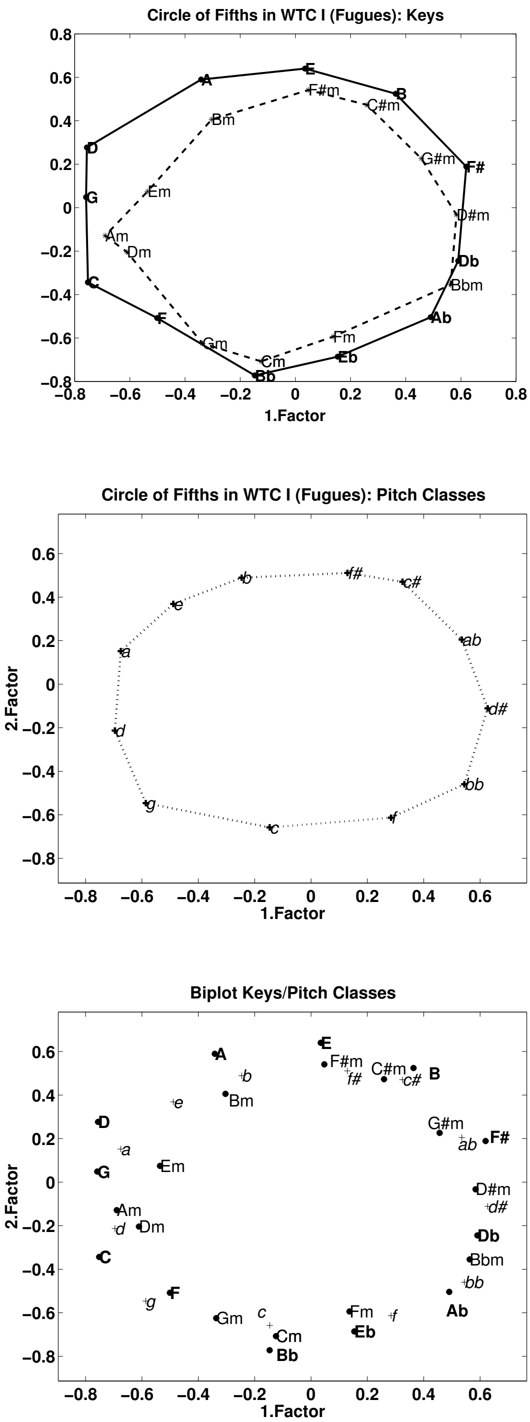

Figure 3: Symbolic durations of keys and pitch classes are derived from the score of the Fugues of Bach's WTC I and then projected onto the factors of correspondence analysis. *Upper*: The emerged circle of fifths is lined out for Major (solid) and minor (dashed) keys ("m" denoting minor). *Middle*: As above pitch classes appear in the order of the circle of fifths. *Lower:* The biplot of keys and pitch classes derived from putting both transparent upper graphs on top of each other. We observe that the pitch classes are close to the corresponding minor keys.

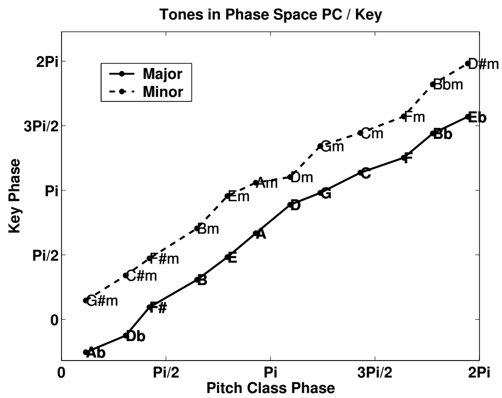

Figure 4: Phase of the circles of fifths described by Major and minor keys (upper Figure 3) relative to the phase of the circle of fifths in pitch classes (middle Figure 3). The graphs describe almost straight parallel lines. The angular co-ordinate of pitch classes and minor keys are "in phase". The angular co-ordinates of pitch classes and Major keys are offset by a constant value slightly less than $\pi/2$.

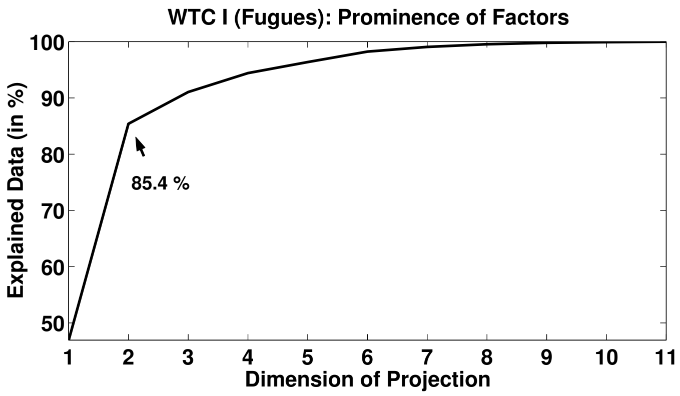

Figure 5: A low-dimensional representation is sufficient for representing the high-dimensional data: For WTC I (Fugues) in score representation the 2-dimensional projection of the overall symbolic duration of keys and pitch classes represents 85.4 % of the variance of the high dimensional vectors. Each of the two most prominent factors have approximately the same singular value.

Major. We conclude that the minor keys induce the circle of fifths in the pitch classes.

In Figure 5 the singular values to the factors are shown. They indicate how much of the variance in the data is captured if correspondence analysis projects the data onto the factors with highest singular values. It is interesting that the explanatory values, e.g., the singular values, of the two most prominent factors in WTC I (Fugues) are almost equal, in contrast to the other singular values, whose explanatory value is much smaller.

Toroidal Model of Inter-Key Relations. Figure 3 displays the projection of keys and pitch classes onto the plane spanned by the two most prominent factors. How can we visualize the projection onto the first four factors? We represent points on each of the planes spanned by the first & second and third & fourth factor, respectively, in polar co-ordinates, i.e., by their polar angle and by their distance to the origin. We then plot their angle in the 1-2-plane against their angle in the 3-4-plane (upper Figure 6). Topologically, we can think of the resulting body as the surface of a torus, which can be parameterized by two angles. Upper Figure 6 can be viewed as a torus if we consider vertical and horizontal periodicity, i.e., we glue together the upper and lower side as well as the right and left side. The three circles of fifths then meander around the torus three times as indicated by the solid, dashed, and dotted lines. In addition to the relationship regarding fifths in upper Figure 6 we see that the projection on the 3. and 4. factor contains information about the inter relation between Major keys and their parallel and relative minor keys.

Consistency with a Geometric Model (Chew, 2000). It is fruitful to compare the toroidal interpretation (upper Figure 6) of the biplot of keys and pitch classes (lower Figure 3) with Chew (2000) (middle Figure 6). In Chew (2000) heterogeneous musical quantities, namely, tones, chords, and keys are embedded in a three-dimensional space, thereby visualizing their inter relations (cf. Appendix 6.2 for technical details). The model is derived from the tonnetz (Euler, 1926; Lewin, 1987). Tones are lined up on a helix along the circle of fifths, circular in the X-Y-plane and elevating in the Z direction. For a triad composed of three tones, construct the triangle whose vertices are given by the tones constituting the triad. Then the triad is represented by the weighted center of gravity of the triangle. In the same way a key is represented as the center of gravity of the triangle whose vertices are the points given by the three main triads (tonic, dominant, subdominant) of the key. We reduce this model to pitch classes and keys, assuming that the intermediate level of chords is implicitly given in the music.

Chew (2000) gives a set of parameters derived by optimization techniques from musically meaningful constraints. We choose a different set of parameters to fit the model to the result of our correspondence analysis as displayed in upper Figure 6. (Cf. Appendix 6.2 for parameters.)

In order to facilitate comparison we choose a two-dimensional visualization of the three-dimensional model in Chew (2000). The projection of the model onto the X-Y-plane is circular. Therefore we can parameterize it as angle and length. We plot the vertical dimension (the elevation of the helix) versus

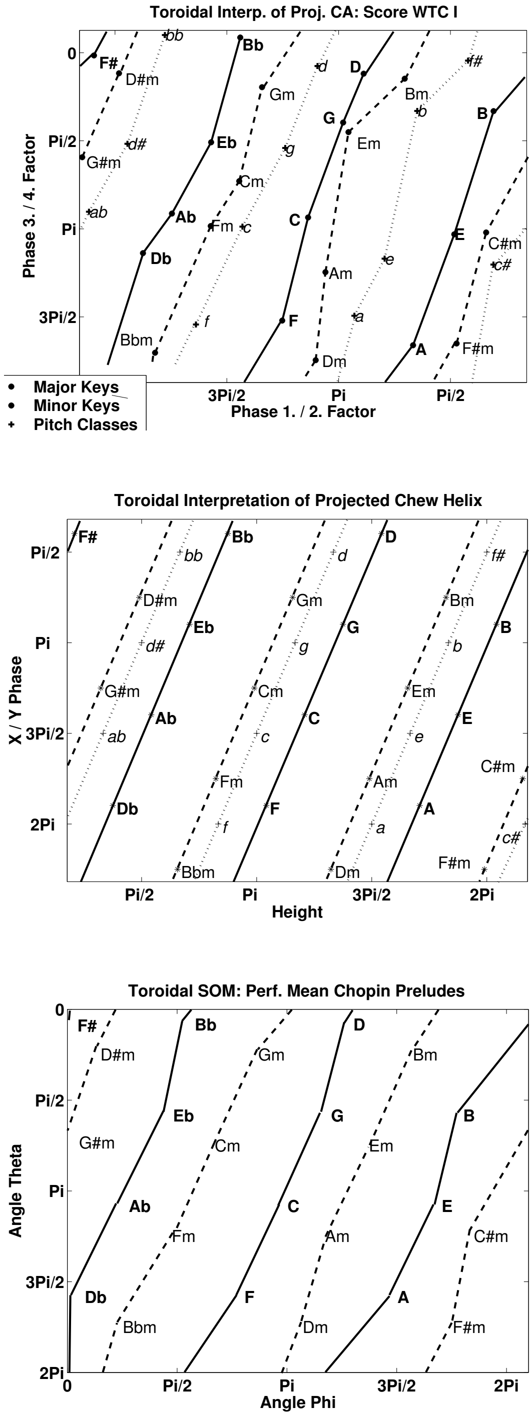

Figure 6: The toroidal interpretation of the projection on the first four factors (*upper*) is consistent with Chew (2000) (*middle*) and Purwins et al. (2000a) (*lower*). A torus can be described in terms of two angles, displayed on the horizontal and vertical axis. Glue the upper/lower and right/left side of the plots together to obtain a torus (cf. text).

the phase angle of the X-Y-plane (middle Figure 6). We interpret the phase angle of the X-Y-plane as the first angle of a torus, and the vertical height in the helix as the second angle of a torus. The helix is mapped on the surface of a torus by applying modulo $12h$ to the height. Here h is the distance on the vertical co-ordinate of two successive tones in the circle of fifths. We observe that upper and middle Figure 6 are very similar: The circles of fifths in Major and minor keys and in pitch classes curl around the torus three times. The only difference is that in the toroidal model derived from correspondence analysis Major keys and their relative minor keys are nearby, whereas in middle Figure 6 Major keys are closer to their parallel minor keys.

Consistency with a Cognitive Model (Purwins et al., 2000a). A very simple listener model comprises the following five stages:

1. Frequency analysis with uniform resolution on a logarithmic scale (constant Q transform of Brown (1991))
2. Compression into pitch class profiles by octave identification
3. Averaging of profiles across each piece
4. Generation of a reference set of profiles, one for each Major and minor key
5. Spatial arrangement of the reference set on a toroidal self-organizing feature map (Purwins et al., 2000b; Kohonen, 1982).

In this scheme, stage 1 can be considered a coarse model of auditory periphery. Stage 5 may be seen as a rough model of cortical feature maps (Obermayer et al., 1990). The constant Q transform is calculated from a digitized 1933/34 recording of Chopin's Préludes Op. 28 performed by Alfred Cortot. The average cq-profiles for each single prelude are used as a training set for a toroidal self-organizing feature map (Purwins et al., 2000b; Kohonen, 1982). Again the resulting configuration (lower Figure 6) shows the circle of fifths and closely resembles the other configurations in Figure 6.

3.2 Circle of Fifths in Performance

We choose a recording of Glenn Gould playing the Preludes and Fugues of Bach's WTC, Book II. We calculate accumulated cq-profiles (Purwins et al., 2000b) from the set of the 24 Fugues of WTC II in all Major and minor keys (cf. Figure 1). Instead of containing frequencies of co-occurrences (cf. Table above) or symbolic durations (cf. Section 3.1 and Figure 3), the co-occurrence table now consists of the intensities of each pitch class accumulated for each of the 24 Fugues in all 24 Major and minor keys (Figure 1).

Pitch classes are represented by 24-dimensional key intensity vectors. In the same way as in Section 3.1, in correspondence analysis a singular value decomposition is performed yielding the factors as a new co-ordinate system. As in middle Figure 3, the pitch class vectors are projected onto a two-dimensional plane, spanned by the two most prominent factors. The circle of fifths evolves in pitch classes embedded in the keyspace in performance data as well. The two factors of performed WTC II (lower Figure 7) capture an even higher percentage (88.54 %) of the variance of the data, than those for the score data of WTC I (cf. Figure 5). Both factors are high and almost equal. Therefore

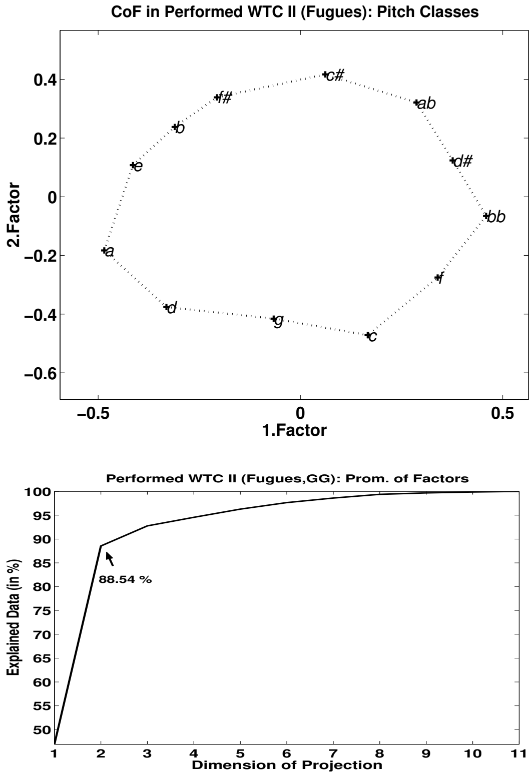

Figure 7: The circle of fifths (lined out) appears also in performed WTC (*upper*). The analyzed data are the overall intensities of pitch classes in the Fugues of Bach's WTC II in the recording of Glenn Gould shown in Figure 1. The same procedure as in Figure 3 (middle) is applied to project the 24-dimensional pitch class vectors onto a two-dimensional plane, spanned by the two most prominent factors. These two factors of performed WTC II capture an even higher percentage (88.54 %, *lower*) of the variance of the data than those for the score data of WTC I (cf. Figure 5).

the two–dimensional projection appears to be a very appropriate representation of pitch classes.

Comparisons have been made with other cycles of musical pieces like Chopin's Préludes Op. 28 and Hindemith's "ludus tonalis": In these cycles one singular value alone is by far most prominent. That means that key frequency space as well as pitch class space can be reduced to one dimension still being able to explain the majority of the data.

4 Stylescapes Based on Key Preference

In the following experiment the interplay of key preference and composer, rather than the interplay of key duration and pitch class duration is considered. For seven composers the frequencies are provided, how many pieces each composer wrote in each of the 24 Major and minor keys. We will discuss key preference in the light of key character and display the relations between different composers and between composers and keys.

Key Preference Statistics. For each key and for each composer the cooccurrence table now contains the number of pieces written in that particular key by that particular composer. We identify enharmonically equivalent keys. Key preference statistics is counted in the following composers: J. S. Bach (JSB, only works for keyboard), L. v. Beethoven (LvB), J. Brahms (JB, non-vocal works), F. Chopin (FC), J. Haydn (JH), W. A. Mozart (WAM), A. Vivaldi (AV). If not stated otherwise, all works of the composer are considered, provided they contain the key name in the title of either the entire work or of single pieces, in case the work consists of a cycle of several pieces. For instance, a sonata in C–Major is accounted for once, but WTC, Book I is accounted for 24 times. These key preference statistics (Figures 8 and 9) were generated from complete work lists of these composers found on the Internet. Auhagen (1983) counts the keys of every single movement in a set of works by Chopin. Enharmonically equivalent keys are counted separately. The most frequently used key is A♭–Major in our statistics as well as in Auhagen (1983). Also the six most frequent keys are the same in our statistics and in Auhagen (1983).

Key Character and Key Statistics. Key character is determined by several factors. One is the complex aspect of keyboard tuning. The composers considered here strongly differ in their preference for A♭–Major. Exemplarily we will inspect the key character of this key and consider its impact on key preference.

In mean tone tuning w.r.t. C, the wolf fifth g♯ – e♭ sounds very rough, since this fifth is 35.7 Cent higher than the just fifth (Meister (1991) p. 81 cited in Grönewald (2003)). Bach used Neidhardt's tuning (Lindley, 2001). Even though in Neidhardt's tuning the wolf is eliminated, Bach used A♭–Major and g♯–minor only in WTC. Bach's avoidance of these keys could be due to the reminiscence that these keys would have sounded odd in a tuning with the wolf on a♭/g♯.

On the other hand A♭–Major was the favored key of Chopin. What did make Chopin prefer this key? Did A♭–Major have a special key character for Chopin? Lindley (2003) points out "the tradition of tender affects" for this key

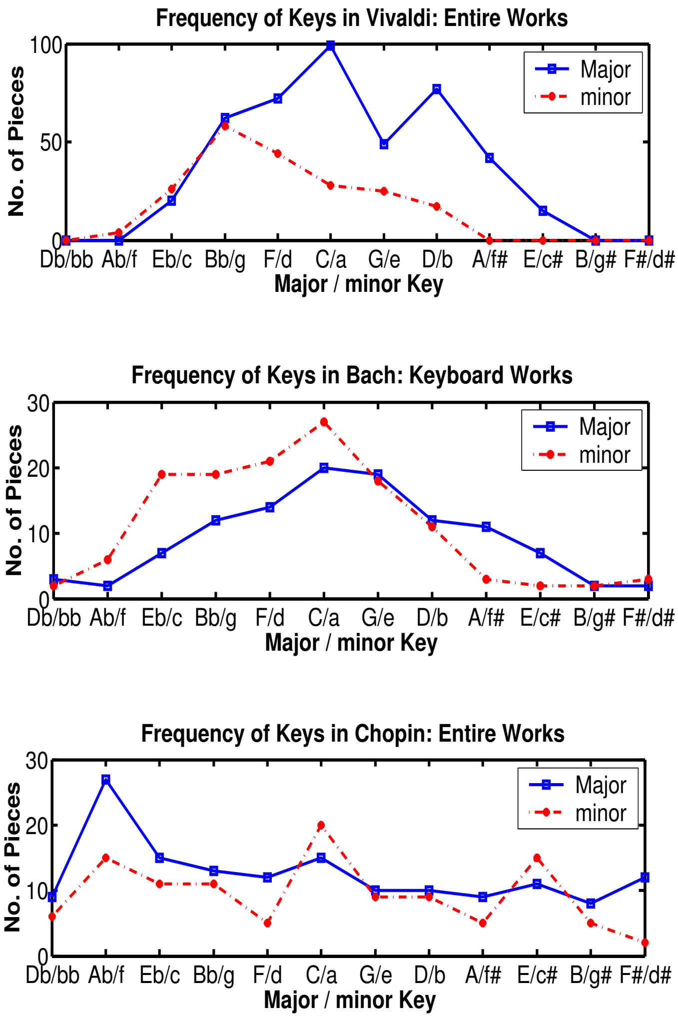

Figure 8: Key preference statistics in complete works of Vivaldi and Chopin, and the keyboard works of Bach. In this and the two subsequent Figures, small letters indicate minor keys. Vivaldi prefers C–Major and D–Major, Bach prefers a–minor, c–minor, and C–major. Some rare keys are only used in the WTC: F♯–Major,B–Major, A♭–Major, c♯–minor, g♯–minor, b♭–minor. 45.5% of the pieces are written in a Major key. The most frequent keys in Chopin are A♭–Major, a–minor, C–Major, E♭–Major, and c♯–minor. There are only two pieces in d♯–minor. 57% of the pieces are in Major.

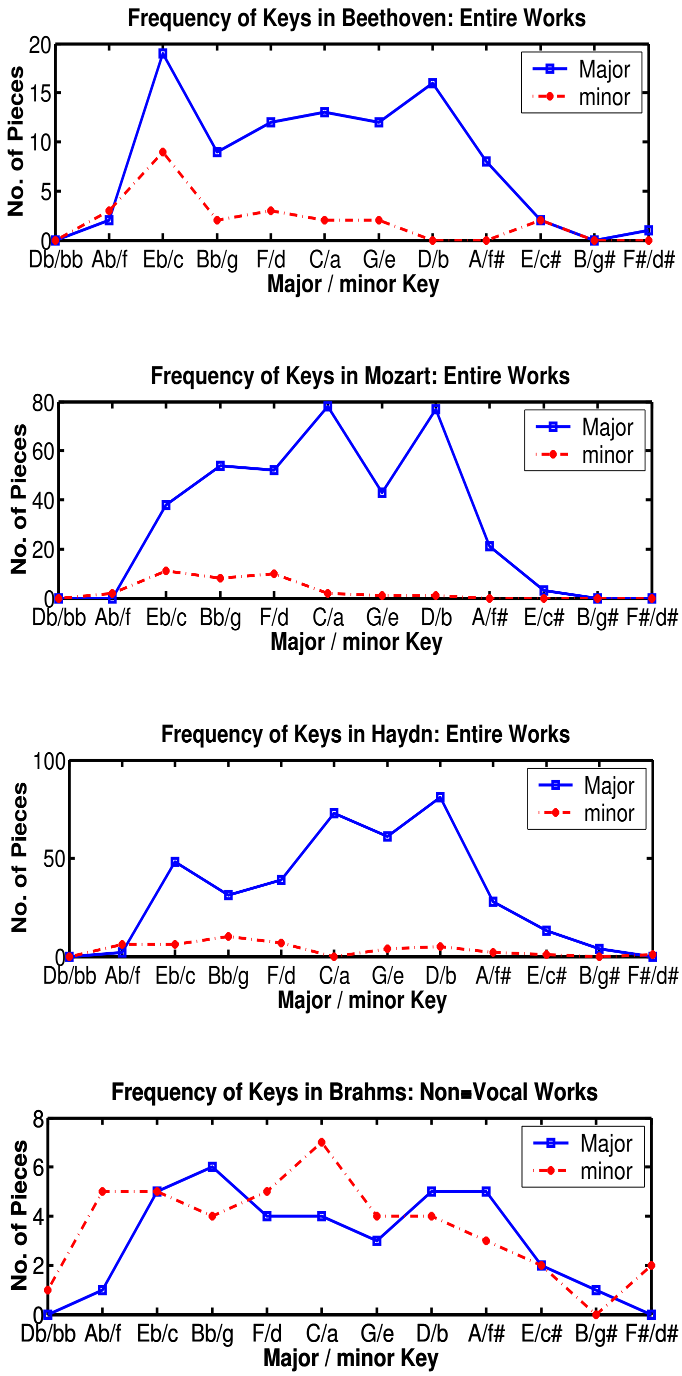

Figure 9: Beethoven prefers the "heroic" E♭–Major and D–Major. As in Vivaldi (cf. Figure 8) the key preference of Mozart and Haydn takes the "Cologne Dome" shape, favoring C–Major and D–Major. 90% of the keys are in Major.

which may have appealed to Chopin's character. Chopin used equal temperament. So all intervals relative to the tonic were the same for all Major keys. For Chopin the key character may have been influenced by 18th century keyboard tuning, that preceded equal temperament. In tunings common at time when "Clementi was flourishing and Beethoven was not yet deaf", A♭–Major had characteristic differences to tunings of keys with none or only few accidentals, e. g. C–Major. First the Major third a♭-c in A♭–Major is stretched compared to c-e in C-Major, second the leading note - tonic step g-a♭ is squeezed compared to b-c. Therefore this tuning endows A♭–Major with a more nervous character and C–Major with a more solid one. [2]

Analysis. Correspondence analysis is performed on the key preferences of the seven composers. It noteworthy that correspondence analysis contributes to the varying amount of data available for the different composers. E. g. low number of available pieces for Brahms is not as important for correspondence analysis as the big number of pieces by Haydn. The projection of the composers into the 2-dimensional plane spanned by the two most prominent factors provides a stylescape: stylistically related composers are close to each other on this plane (cf. Figure 4). In the biplot composers/keys in Figure 4 composers are related to keys: Due to their shared "Cologne Dome" preference for C–Major and D–Major, Haydn and Mozart are very close and Vivaldi is not so far off. Beethoven is close to his favored E♭–Major and near to Haydn and Mozart. Brahms and Bach are positioned in their favored minor keys. Chopin maintains his outlier position due to the outlier position of his favored $A\flat$–Major key. The explanatory value for the first (63.23 %) and the first two factors (88 %) in correspondence analysis is high. The most important factor is very dominant.

5 Discussion

In this paper we have shown how meaningful parameters in the complex structure of music can be visualized, by this revealing the inter relations of music looked upon in the perspective of a certain parameter. To demonstrate the high potential of this approach we have given examples in the domain of inter-key relations based on the perspective of looking at the frequency of pitch class usage and in the domain of stylistic categorization based on a view of the key preference of the different composers. The benefit of the method reveals since the approach is simple but non the less does require almost no assumptions, neither musical knowledge, nor special artificial data. The emergence of the circle of fifths has been observed in previous work on cognitive models. In Leman (1995) artificially generated cadential chord progressions constructed from Shepard tones are used as training data. Purwins et al. (2000a) used overall averaged digitized sound samples (Chopin's Préludes op. 28) for training. In contrast, in the present work we used accumulated vectors of each single digitized recording of the pieces in WTC as training vectors. In both Leman (1995) and Purwins et al. (2000a) the circular key structure is implicitly stipulated by the training of a *toroidal* self-organizing feature map. In the

2 The arguments in this subparagraph are from Lindley (2001, 2003) and from personal communication with Mark Lindley.

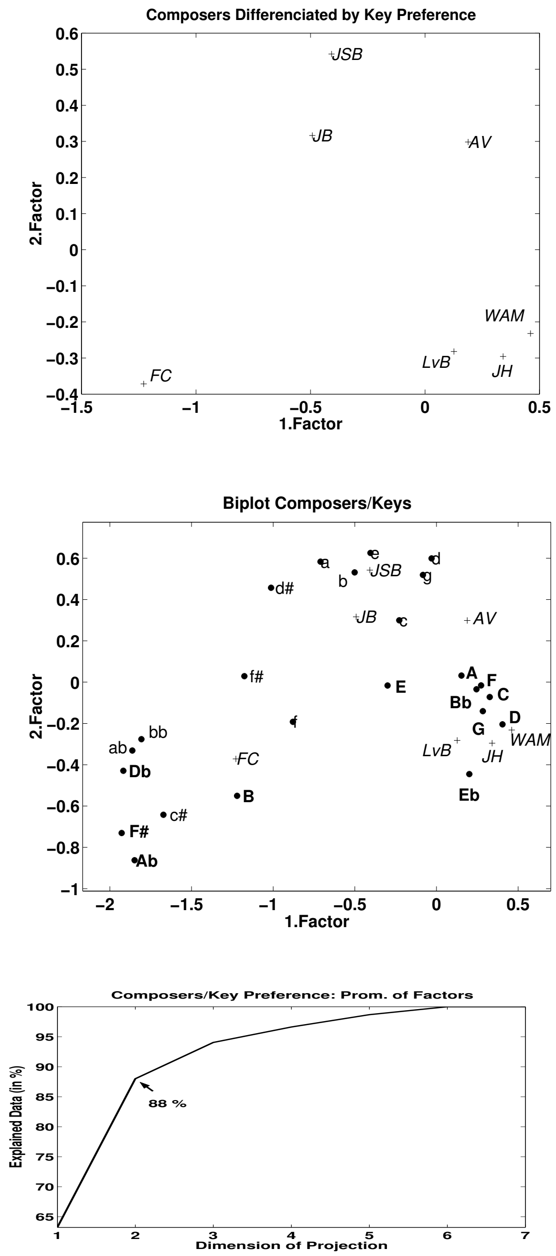

Figure 10: Based on the key preference profiles (cf. Figures 8 and 9) for different composers the stylescape (upper) and the biplot of composers/keys (middle) with associated singular values are shown. Capital letters indicate Major keys, small ones minor keys, italics indicate composers. *Upper:* Haydn (JH) and Mozart (WAM) are very close. Beethoven (LvB) is nearby. Brahms (JB) and Bach (JSB) can be considered a group. Chopin (FC) is an outlier. *Middle:* In the biplot composers/keys we observe that the Viennese classic (JH, WAM, LvB) gather in the region of D–Major and G–Major. Chopin (FC) maintains his outlier position due to the distant position of his favored A♭–Major key. *Lower:* The explanatory values for the first (63.23 %) and the two most prominent factors (88 %) in correspondence analysis are high.

simulation discussed here the circularity emerges from the data alone, without an implicit assumption of periodicity in the model. In this sense, our analysis can be viewed as discovering a model of circular structure rather than merely fitting such a model.

The available data has not been exhaustively analyzed. Projections to different sub-planes could be explored and interpreted. The method can be used to model an experienced listener exposed to a new piece of music, and the listening experience in the context of a body of reference pieces. In correspondence analysis this would correspond to embedding the new pieces in a co-ordinate system obtained from analyzing the reference data. As an example, Bach's WTC has been used to generate a tonal co-ordinate system which then served to embed a number of other works including the Chopin's Préludes Op. 28, Alkan's Préludes, Scriabin's Préludes, Shostakovich's Préludes, and Hindemith's "ludus tonalis". In this way the method can be used to model how a listener who is familiar with Bach's WTC would perceive these keys and pitches in these more recent works. In addition, concepts of inter-key relations underlying Hindemith and Scriabin may be discovered.

We would like to emphasize that the use of correspondence analysis is by no means limited to tonality analysis. The method is a universal and practical tool for discovering and analyzing correspondences between various musical parameters that are adequately represented by co-occurrences of certain musical events or objects. Examples include pitch classes, keys, instrumentation, rhythm, composers, and styles. Three-dimensional co-occurrence arrays, for instance of pitch classes, keys, and metric positions can be analyzed. In particular, it seems promising to extend our analysis to temporal transitions in the space of musical parameters.

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We would like to thank Ulrich Kockelkorn for proof reading and thorough advice in correspondence analysis, Mark Lindley and Hans-Peter Reutter for instruction about temperament, and Thomas Noll for inspiring discussions.

6 Appendix

6.1 Technical Details of Correspondence Analysis

In this appendix we provide some more technical details relating singular value decomposition and correspondence analysis (Greenacre, 1984; Kockelkorn, 2000). The following theorem is crucial for analyzing the co-occurrence matrix in correspondence analysis:

Theorem 1 (Generalized Singular Value Decomposition) *Let* $\mathbf{A}$ *be a positive definite symmetric* $m \times m$ *matrix and* $\mathbf{B}$ *a positive definite symmetric* $n \times n$ *matrix. For any real-valued* $m \times n$ *matrix* $\mathbf{F}$ *of rank* d *there exist an* $m \times d$ *matrix* $\mathbf{U} = (\mathbf{u_1}, \ldots, \mathbf{u}_d)$, *a* $d \times n$ *matrix* $\mathbf{V} = (\mathbf{v_1}, \ldots, \mathbf{v}_d)'$ *with* $\mathbf{U}'\mathbf{A}\mathbf{U} = \mathbf{V}'\mathbf{B}\mathbf{V} = \mathbf{I}_d$, *and a diagonal* $d \times d$ *matrix* $\Delta = (\delta_{ij})$ *so that:*

$$\mathbf{F} = \mathbf{U}\Delta\mathbf{V}' = \sum_{k=1}^{d} \delta_{kk}\mathbf{u}_k\mathbf{v}_k'. \tag{6}$$

Cf. Greenacre (1984) for a proof. For $\mathbf{A} = \mathbf{I}_m, \mathbf{B} = \mathbf{I}_n$, Theorem 1 yields the ordinary singular value decomposition. If furthermore $\mathbf{F}$ is symmetric, we get the familiar eigendecomposition.

The columns $\mathbf{u}_k$ of $\mathbf{U}$ can be viewed as the column factors with singular values δ_{kk}. Vice versa the rows $\mathbf{v}_k$ of $\mathbf{V}$ are the row factors with the same singular values δ_{kk}. The magnitude of $\mathbf{F}$ in each of the d dimensions in the co-ordinate system spanned by the factors $\mathbf{u}_k$ is then given by δ_{kk}.

For the matrix of relative frequencies $\mathbf{F}^{\mathcal{P},\mathcal{K}} = (f_{ij}^{\mathcal{P},\mathcal{K}})$ and positive definite diagonal matrices $(\mathbf{F}^{\mathcal{P},\mathcal{P}})^{-1}$ and $(\mathbf{F}^{\mathcal{K},\mathcal{K}})^{-1}$ with the inverted relative frequencies of row and column features, respectively, on their diagonal, Theorem 1 yields:

$$\mathbf{F}^{\mathcal{P},\mathcal{K}} = \mathbf{U}\Delta\mathbf{V}', \tag{7}$$

with

$$\mathbf{U}'(\mathbf{F}^{\mathcal{P},\mathcal{P}})^{-1}\mathbf{U} = \mathbf{V}'(\mathbf{F}^{\mathcal{K},\mathcal{K}})^{-1}\mathbf{V} = \mathbf{I}_d. \tag{8}$$

Defining

$$\mathbf{S} = (s_{ij}) := \Delta\mathbf{V}'\left(\mathbf{F}^{\mathcal{K},\mathcal{K}}\right)^{-1} \tag{9}$$

we get

$$\mathbf{F}^{\mathcal{P}|\mathcal{K}} = \mathbf{F}^{\mathcal{P},\mathcal{K}}\left(\mathbf{F}^{\mathcal{K},\mathcal{K}}\right)^{-1} = \mathbf{U}\Delta\mathbf{V}'\left(\mathbf{F}^{\mathcal{K},\mathcal{K}}\right)^{-1} = \mathbf{U}\mathbf{S}. \tag{10}$$

Taking the i-th column on both sides of Equation 10 we get

$$\mathbf{f}^{\mathcal{P}|\mathcal{K}=i} = \sum_{k=1}^{d} \mathbf{u}_k s_{ki} \tag{11}$$

The profile $\mathbf{f}^{\mathcal{P}|\mathcal{K}=i}$ is described in terms of co-ordinates s_{ki} on the axes $\mathbf{u}_k$. s_{ki} is the projection – in the χ^2-metric – of profile $\mathbf{f}^{\mathcal{P}|\mathcal{K}=i}$ onto the axis $\mathbf{u}_k$.
Vice versa we have

$$\mathbf{F}^{\mathcal{K}|\mathcal{P}} = \mathbf{V}\underbrace{\Delta\mathbf{U}'\left(\mathbf{F}^{\mathcal{P},\mathcal{P}}\right)^{-1}}_{=:\mathbf{Z}=(z_{ij})} = \mathbf{V}\mathbf{Z}. \tag{12}$$

The profile $\mathbf{f}^{\mathcal{K}|\mathcal{P}=j}$ is described in terms of co-ordinates z_{kj} on the axes $\mathbf{v}_k$.

Each key $\mathcal{K} = i$ is given by its pitch class profile $\mathbf{f}^{\mathcal{P}|\mathcal{K}=i}$. In Figure 3 key $\mathcal{K} = i$ is represented by its first two coordinates (s_{1i}, s_{2i}).

Each pitch class $\mathcal{P} = j$ is given by its key profile $\mathbf{f}^{\mathcal{K}|\mathcal{P}=j}$. In Figures 3 and 7, pitch class $\mathcal{P} = j$ is represented by its first two coordinates (z_{1j}, z_{2j}).

6.2 Details of Chew's Model with Choice of Parameters

We use a simplified instance of Chew's more general model (Chew, 2000). It proposes a spatial arrangement such that tones, triads and keys are represented as vectors in three-dimensional space. For $j \in \mathbb{N}$ tones are denoted by $\mathbf{t}(j) \in \mathbb{R}^3$, proceeding in steps of one fifth interval from index j to index $j+1$. We denote major and minor triads by $\mathbf{c}_{\mathrm{M}}(j), \mathbf{c}_{\mathrm{m}}(j) \in \mathbb{R}^3$, and Major and minor keys by $\mathbf{k}_{\mathrm{M}}(j), \mathbf{k}_{\mathrm{m}}(j) \in \mathbb{R}^3$, respectively.

The tones are arranged in a helix turning by $\pi/2$ and rising by a factor of h per fifth:

$$\mathbf{t}(j) = \left(\sin\left(\frac{j\pi}{2}\right), \cos\left(\frac{j\pi}{2}\right), jh\right) \tag{13}$$

Both Major and minor triads are represented as the weighted mean of their constituent tones:

$$\begin{aligned} \mathbf{c}_{\mathrm{M}}(j) &= m_1\mathbf{t}(j) + m_2\mathbf{t}(j+1) + m_3\mathbf{t}(j+4) && (14) \\ \mathbf{c}_{\mathrm{m}}(j) &= m_1\mathbf{t}(j) + m_2\mathbf{t}(j+1) + m_3\mathbf{t}(j-3) && (15) \end{aligned}$$

The keys are represented as weighted combinations of tonic, dominant, and subdominant, with the minor keys additionally incorporating some of the Major chords:

$$\begin{aligned} \mathbf{k}_{\mathrm{M}}(j) &= m_1\mathbf{c}_{\mathrm{M}}(j) + m_2\mathbf{c}_{\mathrm{M}}(j+1) + m_3\mathbf{c}_{\mathrm{M}}(j-1) && (16) \\ \mathbf{k}_{\mathrm{m}}(j) &= m_1\mathbf{c}_{\mathrm{m}}(j) + m_2\left(\frac{3}{4}\mathbf{c}_{\mathrm{M}}(j+1) + \frac{1}{4}\mathbf{c}_{\mathrm{m}}(j+1)\right) && (17) \\ &\quad + m_3\left(\frac{3}{4}\mathbf{c}_{\mathrm{m}}(j-1) + \frac{1}{4}\mathbf{c}_{\mathrm{M}}(j-1)\right). && (18) \end{aligned}$$

We choose the parameters so as to obtain a good fit with the data from Bach's WTC I (Fugues), resulting in the following parameter settings:

$$\mathbf{m} = (m_1, m_2, m_3) = (\frac{1}{2}, \frac{1}{4}, \frac{1}{4}) \text{ and } h = \frac{\pi}{6}. \tag{19}$$

The general model of Chew (2000) allows the weights to be independent from each other for $\mathbf{c}_{\mathrm{M}}, \mathbf{c}_{\mathrm{m}}, \mathbf{k}_{\mathrm{M}}, \mathbf{k}_{\mathrm{m}}$. In this application, weights $\mathbf{m}$ are set equal, across Major and minor chords and keys (cf. Chew (2001)). In our instance of the model, we have only $\mathbf{m}$ and h as free parameters. But we use values different from Chew (2001).

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